Differential Realizations of the Two-Mode Bosonic and Fermionic Hamiltonians: A unified Approach

Hayriye Tütüncüler* and Ramazan Koç†

Department of Physics, Faculty of Engineering University of Gaziantep, 27310 Gaziantep, Turkey

(Dated: February 7, 2008)

A method is developed to determine the eigenvalues and eigenfunction of two-boson 2×2 matrix Hamiltonians include a wide class of quantum optical models. The quantum Hamiltonians have been transformed in the form of the one variable differential equation and the conditions for its solvability have been discussed. We present two different transformation procedure and we show our approach unify various approaches based on Lie algebraic technique. As an application, solutions of the modified Jaynes-Cummings and two-level Jahn Teller Hamiltonians are studied.

PACS numbers:

INTRODUCTION

The algebraic techniques have been proven to be useful in the description of the physical problems in a variety of fields. Recently a new algebraic approach, essentially improving both analytical and numerical solution of the problems, has been suggested and developed for some nonlinear quantum optical systems[1, 2, 3, 4]. Most of such developments are mainly based on linear Lie algebras, but it is evident that there is no physical reason for symmetries to be only linear. Nonlinear Lie algebra techniques and their relations to the nonlinear quantum optical systems have been discussed[5, 6, 7, 8, 9]. In both cases finite part of spectrum of the corresponding Hamiltonian can be exactly obtained in closed forms and these systems are known as quasi-exactly-solvable(QES) termed by Turbiner and Ushveridze[10]. It has been proven that the single boson Hamiltonians also lead to a QES under some certain constraints[11, 12].

In recent years there has been a great deal of interest in quantum optical models which reveal new physical phenomena described by the Hamiltonians expressed as nonlinear functions of Lie algebra generators or boson and/or fermion operators[13, 14, 15, 16, 17]. Such systems have often been analyzed by using numerical methods, because the implementation of the Lie algebraic techniques to solve those problems is not very efficient and most of the other analytical techniques do not yield simple analytical expressions. They require tedious calculations[18, 19, 20].

The aim of this paper is to determine solvability of the two-boson 2×2 matrix Hamiltonians and discuss their applications and possible symmetry groups. The results of our procedure include the solutions of the Hamiltonians that possessing $su(2)$, $su(1,1)$, $Sp(4,R)$, $osp(2,1)$ and $osp(2,2)$ symmetries. The procedure presented in this paper also leads to the constructions of the nonlinear Lie algebras[21]. The Hamiltonians discussed here are not only mathematically interesting but they have potential interest in physics[22, 23, 24]. In order to keep our discussion simple we concentrate our attention to the solution of the Hamiltonians that include two boson and one fermion. Therefore we provide a first step toward the extension of the technique to the solution of the multi boson fermion systems.

The paper is organized as follows: In section 2, we construct a Hamiltonian including two boson operators in an arbitrary order and one fermion operator. Solutions of the Hamiltonians by using the invariance of the number operator have been discussed. As a practical example, solution of the Modified Jaynes-Cummings and two level Jahn Teller Hamiltonians have been obtained. In section 3 we present two transformation procedure that is appropriate to determine the conditions of the (quasi)exact solvability of the Hamiltonian. In section 4 we discuss the symmetry properties of the Hamiltonian. The results of our discussion are that the Hamiltonian contains various Lie (super)algebras, namely, $su(2), su(1,1), Sp(4,R), osp(2,1)$ and $osp(2,2)$. We also point out that nonlinear Lie algebras can be constructed as spectrum generating algebras. As an application we visit Jaynes-Cummings Hamiltonian including Kerr nonlinearity. The importance of our approach is that it provides a unification of the various approaches. Finally, in section 5 we comment on the validity of our method and suggest the possible extensions of the problem.

TWO BOSON ONE FERMION HAMILTONIAN AND ITS DIFFERENTIAL REALIZATION

Two mode bosonic 2×2 matrix Hamiltonians play an important role in nonlinear quantum optical systems. The Hamiltonians of such systems can be generalized as follows:

$$H = \sum_{k_i} \alpha_{k_1,k_2,k_3,k_4} B^k + \sigma_0 \sum_{\ell_i} \beta_{\ell_1,\ell_2,\ell_3,\ell_4} B^\ell +$$

$$\sigma_+ \sum_{m_i} \gamma_{m_1,m_2,m_3,m_4} B^m + \sigma_- \sum_{n_i} \delta_{n_1,n_2,n_3,n_4} B^n. \tag{1}$$

The bosonic operator B^v reads as follows:

$$B^v = (a_1^+)^{v_1} (a_1)^{v_2} (a_2^+)^{v_3} (a_2)^{v_4} \tag{2}$$

The constants $\alpha_i, \beta_i, \gamma_i$ and δ_i are related to the parameters of the physical Hamiltonian and k_i, ℓ_i, m_i and n_i determine order of the interaction. The boson creation, a_1, a_2, and annihilation a_1^+, a_2^+ operators obey the usual commutation relations

$$[a_i, a_i] = [a_i^+, a_i^+] = 0, \quad [a_i, a_j^+] = \begin{cases} 1, i = j \\ 0, i \neq j \end{cases} \tag{3}$$

and $\sigma_{\pm,0}$ are Pauli matrices:

$$\sigma_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \sigma_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{4}$$

The number operator of such system can be expressed in the form

$$N = sa_1^+ a_1 + pa_2^+ a_2 + r\sigma_0 \tag{5}$$

and satisfy the commutation relations

$$[N, a_1^+] = sa_1^+, \quad [N, a_1] = -sa_1, \quad [N, a_2^+] = pa_2^+, \quad [N, a_2] = -pa_2$$
$$[N, \sigma_\pm] = \pm 2r\sigma_\pm \quad [N, \sigma_0] = 0. \tag{6}$$

The action of the number operator on the state $|n_1, n_2\rangle$ is given by

$$N |n_1, n_2\rangle = (sn_1 + pn_2 + r) |n_1, n_2\rangle \tag{7}$$

The eigenvalue equation (6) leads to the following solution:

$$|n_1, n_2\rangle = (a_1^+)^j \phi_1 \left(a_2^+ (a_1^+)^{-\frac{p}{s}}\right) |\uparrow\rangle + (a_1^+)^{j+\frac{2r}{p}} \phi_2 \left(a_2^+ (a_1^+)^{-\frac{p}{s}}\right) |\downarrow\rangle. \tag{8}$$

where $|\uparrow\rangle$ and $|\downarrow\rangle$ are up and down states and j is given by

$$j = n_1 + \frac{p}{s} n_2. \tag{9}$$

If N and H commute, the function (8) is also eigenfunction of the H. Therefore it is worth to seek the conditions for the commutation of H and N. This can easily be done by using the commutation relations (3) and (6) and we obtain the following relation

$$[N, H] = \sum_{k_i} [s(k_1 - k_2) + p(k_3 - k_4)] \alpha_{k_1 k_2 k_3 k_4} B^k +$$

$$\sigma_0 \sum_{\ell_i} [s(\ell_1 - \ell_2) + p(\ell_3 - \ell_4)] \beta_{\ell_1 \ell_2 \ell_3 \ell_4} B^\ell +$$

$$\sigma_+ \sum_{m_i} [s(m_1 - m_2) + p(m_3 - m_4) + r] \gamma_{m_1 m_2 m_3 m_4} B^m + \tag{10}$$

$$\sigma_- \sum_{n_i} [s(n_1 - n_2) + p(n_3 - n_4) - r] \delta_{n_1 n_2 n_3 n_4} B^n$$

Then the constant of motion N and H commute when the fallowing set of equation is satisfied

$$
\begin{aligned}
s(k_1 - k_2) + p(k_3 - k_4) &= 0 \\
s(\ell_1 - \ell_2) + p(\ell_3 - \ell_4) &= 0 \\
s(m_1 - m_2) + p(m_3 - m_4) + 2r &= 0 \\
s(n_1 - n_2) + p(n_3 - n_4) - 2r &= 0.
\end{aligned}
\tag{11}
$$

Now we demonstrate application of the procedure on a physical example. We study modified Jaynes-Cummings Hamiltonian and this give us an opportunity to test our approach because those Hamiltonians have been studied in literature.

The modified Jaynes-Cummings Hamiltonian

The modified Jaynes-Cummings Hamiltonian have been constructed to investigate single two level atom placed in the common domain of two cavities interacting with two quantized modes. It is given by[25]:

$$
H = \omega(a_1^+ a_1 + a_2^+ a_2) + \frac{\omega_0}{2}\sigma_0 + \lambda_1(a_1\sigma_+ + a_1^+\sigma_-) + \lambda_2(a_2\sigma_+ + a_2^+\sigma_-).
\tag{12}
$$

When the parameters

$$
\alpha_{i,j,k,l} = \beta_{i,j,k,l} = \gamma_{i,j,k,l} = \delta_{i,j,k,l} = 0
\tag{13}
$$

except that

$$
\begin{aligned}
\alpha_{1,1,0,0} &= \alpha_{0,0,1,1} = \omega, \beta_{0,0,0,0} = \frac{\omega_0}{2}, \\
\gamma_{0,1,0,0} &= \delta_{1,0,0,0} = \lambda_1, \gamma_{0,0,0,1} = \delta_{0,0,1,0} = \lambda_2
\end{aligned}
\tag{14}
$$

then the Hamiltonians (1) and (12) are identical. The condition (11) is satisfied when $s = p = 2r$. Before going further in the following we use the Bargmann-Fock representation, where creation and annihilation operators are replaced by multiplication and differentiation operators:

$$
a_i^+ = z_i, \quad a_i = \frac{d}{dz_i}
\tag{15}
$$

with respect to complex variable z_i. The eigenfunction (8) is of the form

$$
|n_1, n_2\rangle = (z_1)^j \phi(x) |\uparrow\rangle + (z_1)^{j+1} \phi(x) |\downarrow\rangle
\tag{16}
$$

where $x = (z_1)^{-1} z_2$. The solution of this system describes a quantum mechanical state of H provided that $\phi(x)$ belong to the Bargmann-Fock space. The scalar product should be complete and normalizable,

$$
\int \overline{\phi(x)}\phi(x)e^{\int W(x)dx} d(Rex)d(Imx) < \infty
\tag{17}
$$

where $W(x)$ is the weight function. The eigenvalue equation of the modified Jaynes-Cummings Hamiltonian can be written as

$$
H |n_1, n_2\rangle = E |n_1, n_2\rangle
\tag{18}
$$

Insertion of (15) and (16) into (18) yield the following two set of differential equation

$$
\begin{aligned}
\left[j\omega + \frac{\omega_0}{2} - E\right]\phi_1(x) + (j+1)\lambda_2\phi_2(x) + (\lambda_2 - x\lambda_1)\frac{d\phi_2(x)}{dx} &= 0 \\
\left[(j+1)\omega - \frac{\omega_0}{2} - E\right]\phi_2(x) + (\lambda_1 + x\lambda_2)\phi_1(x) &= 0
\end{aligned}
\tag{19}
$$

Bargmann-Fock space solution of the (19) can easily be obtained and they are given by

$$
\begin{aligned}
\phi_1(x) &= C_0 \left(\lambda_2 - x\lambda_1\right)^{j-n+1} \left(\lambda_1 + x\lambda_2\right)^{n-1} \\
\phi_2(x) &= C_1 \left(\lambda_2 - x\lambda_1\right)^{j-n+1} \left(\lambda_1 + x\lambda_2\right)^{n}
\end{aligned}
\tag{20}
$$

where n is an integer and eigenvalues of the Hamiltonian is given by

$$
E = \frac{1}{2} \left((2j+1)\omega \pm \sqrt{4n(\lambda_1^2 + \lambda_2^2) + (\omega_0 - \omega)^2} \right).
\tag{21}
$$

Consequently we have obtained exact result for the eigenvalues of the modified Jaynes-Cummings Hamiltonian. The same result have been obtained in[25], in the framework of the $su(2)$ algebra. The procedure given here can be applied to obtain eigenfunction and eigenvalues of the various physical Hamiltonians. In the following example we consider the solution of the Jahn-Teller distortion problem.

Two-Level Jahn-Teller Distortion Problem

The well-known form of the JT Hamiltonian describing a two-level fermionic subsystem coupled to two boson modes has been given by [26]:

$$
H = a_1^+ a_1 + a_2^+ a_2 + 1 + (\frac{1}{2} + 2\mu)\sigma_0 + 2\kappa[(a_1 + a_2^+)\sigma_+ + (a_1^+ + a_2)\sigma_-].
\tag{22}
$$

Our task is now to demonstrate the Hamiltonian (22) can be solved in the framework of the procedure given previous section. It will be shown that our approach relatively very simple when compared previous approaches. The Hamiltonians (22) and (1) are identical, when the parameters are constrained to:

$$
\begin{aligned}
\alpha_{1,1,0,0} &= \alpha_{0,0,1,1} = \alpha_{0,0,0,0} = 1, \quad \beta_{0,0,0,0} = (\frac{1}{2} + 2\mu), \\
\gamma_{0,1,0,0} &= \delta_{1,0,0,0} = \gamma_{0,0,1,0} = \delta_{0,0,0,1} = 2\kappa
\end{aligned}
\tag{23}
$$

otherwise

$$
\alpha_{i,j,k,l} = \beta_{i,j,k,l} = \gamma_{i,j,k,l} = \delta_{i,j,k,l} = 0
\tag{24}
$$

The condition (11) is satisfied when $s = -p = 2r$. In the Bargmann-Fock space the eigenfunction (8) takes the form

$$
|n_1, n_2\rangle = (z_1)^j \phi_1(x) |\uparrow\rangle + (z_1)^{j-1} \phi_2(x) |\downarrow\rangle .
\tag{25}
$$

where $x = z_1 z_2$ and $j = n_1 - n_2$. Since number operator N and Hamiltonian (22) commute, they have the same eigenfunction. Substituting (25) into (22) we obtain the following set of equation:

$$
x\phi_1'(x) + \kappa x \phi_2'(x) + \frac{1}{4}(3 - 2E + 2j + 4\mu)\phi_1(x) + \kappa(1 + j + x)\phi_2(x) = 0
\tag{26}
$$

$$
\kappa\phi_1'(x) + x\phi_2'(x) + \frac{1}{4}(3 - 2E + 2j - 4\mu)\phi_2(x) + \kappa\phi_1(x) = 0
\tag{27}
$$

These coupled differential equations represent the Schrödinger equation of the $E \otimes \epsilon$ Jahn-Teller system in the Bargmann's Hilbert space. The equation is quasi-exactly-solvable and success of our analysis leads to the solution of the various quantum optical systems. The physical systems described by the differential equations (26, 27) are discussed in[27].

The validity of the procedure depends on the choice of the α_i, β_i, γ_i and δ_i. One can easily be obtain various physical Hamiltonians by appropriate choice of α_i, β_i, γ_i and δ_i and by considering the conditions given in (11).

TRANSFORMATION OF THE OPERATORS

In this section we discuss transformation of the Pauli matrices and boson operators. These transformations plays a key role to construct (quasi)exactly solvable 2×2 matrix Hamiltonians. The transformation can be done by introducing the following similarity transformation induced by the metric

$$S = (a_2^+)^{ca_1^+ a_1 + d\sigma_+\sigma_-} \tag{28}$$

where c and d are constants. Since a_1, and a_2 commute and $\sigma_{\pm,0}$ also commute with the bosonic operators, the transformation of a_1 and a_1^+ under S can be obtained by writing $a_2^+ = e^b$, with $[a_1, b] = [a_1^+, b] = 0$,

$$\begin{aligned} Sa_1 S^{-1} &= a_1(a_2^+)^{-c} \\ Sa_1^+ S^{-1} &= a_1^+(a_2^+)^c \end{aligned} \tag{29}$$

the transformation of a_2 and a_2^+ is as follows

$$\begin{aligned} Sa_2 S^{-1} &= a_2 - (ca_1^+ a_1 + d\sigma_+\sigma_-)(a_2^+)^{-1} \\ Sa_2^+ S^{-1} &= a_2^+ \end{aligned} \tag{30}$$

and the transformations of the $\sigma_\pm$ are given by

$$S\sigma_\pm S^{-1} = \sigma_\pm(a_2^+)^{\pm d}. \tag{31}$$

Before constructing one variable (quasi)exactly solvable differential equation of the Hamiltonian (1) under the transformations of the bosonic and fermionic operators by S, let us consider the other transformation operator:

$$T = (a_2)^{\varepsilon a_1^+ a_1 + \eta\sigma_+\sigma_-} \tag{32}$$

where ε and η are constants. By using the similar arguments given in the previous transformation operations one can easily obtain the following transformations:

$$\begin{aligned} Ta_1 T^{-1} &= a_1(a_2^+)^{-\varepsilon} \\ Ta_1^+ T^{-1} &= a_1^+(a_2)^{\varepsilon} \\ Ta_2 T^{-1} &= a_2 \\ Ta_2^+ T^{-1} &= a_2^+ + (\varepsilon a_1^+ a_1 + \eta\sigma_+\sigma_-)(a_2)^{-1} \\ T\sigma_\pm T^{-1} &= \sigma_\pm(a_2^+)^{\pm\eta}. \end{aligned} \tag{33}$$

These transformations leads to the construction of the various differential realizations of (1), depending on the choice of c, n, ε and η. In the following we discuss the possible forms of the Hamiltonian (1) and their solutions.

S-TRANSFORMED HAMILTONIAN

In this section we discuss the solvability of the Hamiltonian(1). The number operator N describe the states of the corresponding Hamiltonian. The transformation of N under the operator S is given by

$$N' = SNS^{-1} = (s - pc)a_1^+ a_1 + pa_2^+ a_2 + (2r - pd)\sigma_+\sigma_- - r \tag{34}$$

The Hamiltonian (1) is characterized by the total number of a_1 and a_2 bosons when the number operator N commute with the whole Hamiltonian. In the transformed case it is only the number of a_2 bosons that characterize the system under the condition $c = s/p$ and $d = 2r/p$. When the representation is characterized by a fixed number $a_2^+ a_2 = j$, then the transformed form the Hamiltonian can be expressed as one boson operator a_1, under the condition (11). The

transformed form of the Hamiltonian (1) can be written as:

$$
\begin{aligned}
\widetilde{H} \;=\; SHS^{-1} =& \sum_{k_i} \alpha_{k_1,k_2,k_4}(a_1^+)^{k_1}(a_1)^{k_2}(j - \frac{s}{p}a_1^+ a_1 - \frac{2r}{p}\sigma_+\sigma_-)^{k_4} \\
& \sigma_0 \sum_{\ell_i} \beta_{\ell_1,\ell_2,\ell_4}(a_1^+)^{\ell_1}(a_1)^{\ell_2}(j - \frac{s}{p}a_1^+ a_1 - \frac{2r}{p}\sigma_+\sigma_-)^{\ell_4} + \\
& \sigma_+ \sum_{m_i} \gamma_{m_1,m_2,m_4}(a_1^+)^{m_1}(a_1)^{m_2}(j - \frac{s}{p}a_1^+ a_1 - \frac{2r}{p}\sigma_+\sigma_-)^{m_4} + \\
& \sigma_- \sum_{n_i} \delta_{n_1,n_2,n_4}(a_1^+)^{n_1}(a_1)^{n_2}(j - \frac{s}{p}a_1^+ a_1 - \frac{2r}{p}\sigma_+\sigma_-)^{n_4}.
\end{aligned}
\tag{35}
$$

Note that k_3, ℓ_3, m_3 and n_3 are eliminated by using the conditions given in(11) The difference between (1) and (35) is that while in the first the total number of a_1 and a_2 bosons characterize the the system, in the later it is only the number of a_2 bosons that characterize the system. Therefore the representation is characterized by a fixed number j and in (35), the Hamiltonian is expressed in terms of one boson operator a_1. The transformed Hamiltonian $\widetilde{H}$, in the Bargmann-Fock space, which play an important role in the quasi-exact solution of the equation (1). It can be transformed in the form of the one dimensional differential equations in the Bargmann-Fock space when the boson operators are realized as

$$
a_1 = \frac{d}{dx}, \quad a_1^+ = x.
\tag{36}
$$

The basis function of the primed generators of the system is two component spinor;

$$
P_{n,m}(x) = \left(\begin{array}{c} x^0, x^1, \cdots, x^n \\ x^0, x^1, \cdots, x^m \end{array} \right).
\tag{37}
$$

Action of (35) on the (37), in the Bargmann-Fock space can be written as:

$$
\widetilde{H}P_n(x) = \sum P_{n,m}(E)(x^n, x^m)
\tag{38}
$$

The wavefunction is itself the generating function of the energy polynomials. The eigenvalues are then produced by the roots of such polynomials. If the $E_{n,m}$ is a root of the polynomial $P_{n+1,m+1}(E)$, the series (38) terminates and $E_{n,m}$ belongs to the spectrum of the corresponding Hamiltonian. The eigenvalues are then obtained by finding the roots of such polynomials.

T-TRANSFORMED HAMILTONIAN

The constant of motion N characterize the system can be transformed, by the operator T is given by

$$
N' = TNT^{-1} = (s + p\varepsilon)a_1^+ a_1 + pa_2^+ a_2 + p\eta\sigma_+\sigma_- + r.
\tag{39}
$$

The Hamiltonian (1) can be characterized, in the transformed case, $\varepsilon = -s/p$ and $\eta = -2r/p$. Thus according to (39) the representation is characterized by a fixed number $a_2^+ a_2 = j - 1$. Therefore the transformed Hamiltonian includes one boson operator a_1, when the condition (11) is taken into consideration, it can be written as:

$$
\begin{aligned}
H' \;=\; THT^{-1} =& \sum_{k_i} \alpha_{k_1,k_2,k_4}(a_1^+)^{k_1}(a_1)^{k_2}(j - \frac{k_4}{k_3} - \frac{s}{p}a_1^+ a_1 - \frac{2r}{p}\sigma_+\sigma_-)^{k_4} \\
& \sigma_0 \sum_{\ell_i} \beta_{\ell_1,\ell_2,\ell_4}(a_1^+)^{\ell_1}(a_1)^{\ell_2}(j - \frac{\ell_4}{\ell_3} - \frac{s}{p}a_1^+ a_1 - \frac{2r}{p}\sigma_+\sigma_-)^{\ell_4} + \\
& \sigma_+ \sum_{m_i} \gamma_{m_1,m_2,m_4}(a_1^+)^{m_1}(a_1)^{m_2}(j - \frac{m_4}{m_3} - \frac{s}{p}a_1^+ a_1 - \frac{2r}{p}\sigma_+\sigma_-)^{m_4} + \\
& \sigma_- \sum_{n_i} \delta_{n_1,n_2,n_4}(a_1^+)^{n_1}(a_1)^{n_2}(j - \frac{n_4}{n_3} - \frac{s}{p}a_1^+ a_1 - \frac{2r}{p}\sigma_+\sigma_-)^{n_4}.
\end{aligned}
\tag{40}
$$

The Hamiltonian can be expressed as one dimensional differential equation in the Bargmann-Fock space.

In order to obtain exactly or QES Hamiltonians one can use the same basis given in (37). The physical Hamiltonians can be obtained and solved by the choice of the appropriate values of the s and p. Consequently we have obtained two classes of Hamiltonians whose spectrum can be obtained (quasi)exactly. In the following section we discuss the symmetry properties of the general Hamiltonian (1).

DISCUSSION: THE UNIFIED APPROACH

In the previous sections transformations of the general two-mode bosonic 2×2 Hamiltonian were treated and its differential realization in the Bargmann-Fock space have been obtained. In addition, we note that our approach unifies various Lie algebraic methods. The algebraic approach to the finite dimensional part of spectrum consists in expressing the Hamiltonian, in terms of the generators of an Lie algebra. When the Hamiltonian can be written in terms of the Casimir invariants of the algebraic structure then the eigenvalue problem $H\psi = E\psi$ can be solved in the closed form, giving rise to energy formulas[28, 29]. Otherwise, in general, the spectrum of H cannot be calculates in the closed form. If it is written in terms of the bilinear combinations of the generators of the Lie algebra then the eigenvalue problem $H\psi = E\psi$ must be solved numerically or one can obtain quasi exact solutions.

A convenient way to construct a spectrum generating algebra is to introduce the boson representations of the corresponding Lie algebra. Let us turn our attention to the purely bosonic part of the Hamiltonian (1). This can be obtained by setting $\beta = \gamma = \delta = 0$. Under some certain conditions one can obtain and Hamiltonian which include bilinear products of the bosonic operators:

$$a_1^+ a_1, a_2^+ a_2, a_1^+ a_2, a_2^+ a_1 \tag{41}$$

generates the Lie algebra $su(2)$ and one can recast these four generators in a more familiar form by introducing the three generators

$$J_0 = \frac{1}{2} \left(a_1^+ a_1 - a_2^+ a_2 \right), J_+ = a_1^+ a_2, J_- = a_2^+ a_1. \tag{42}$$

The one variable differential realizations of the generators (42) can be obtained by the transformation procedure given in the previous sections and they play an important role to the QES of the Hamiltonian which underlying $su(2)$ symmetry.

The other important Lie algebra that included by the Hamiltonian (1) is the $su(1,1)$ algebra. Bosonic representations of the $su(1,1)$ algebra is given by

$$K_0 = \frac{1}{2} \left(a_1^+ a_1 + a_2^+ a_2 + 1 \right), K_+ = a_1^+ a_2^+, K_- = a_2 a_1. \tag{43}$$

Therefore by the appropriate choice of the parameter $\alpha_{i,j,k,l}$ of (1) one can construct the Hamiltonian possesses the symmetry of the $su(1,1)$. The transformations of the operators of the $su(1,1)$ lead to the one dimensional differential realizations of the algebra. Both in the $su(2)$ and $su(1,1)$ algebras the transformed operators are in the form of the operators of the sl-algebra. One can treat bound state problems by using the $su(2)$ algebra and scattering state problems $su(1,1)$ algebra. Thus one can have transition from one bound state to another and from a scattering state to another. In order to calculate from bound states to scattering states we need a larger algebra[28, 30, 31]. We can construct this algebra by considering the bilinear combinations of the bosonic operators

$$a_1^+ a_1, a_2^+ a_2, a_1^+ a_2, a_2^+ a_1, a_1^+ a_2^+, a_2 a_1, a_1 a_1, a_2 a_2, a_1^+ a_1^+, a_2^+ a_2^+. \tag{44}$$

One can show that these 10 operators close under the sympletic algebra $Sp(4, R)$. The algebra $Sp(4, R)$ contains both bound state algebra $su(2)$ and scattering state algebra $su(1,1)$. Therefore the algebra provides a unified treatment within both bound and scattering states. Its generators can connect all states in the same potential. It thus provide a unified approach to the one dimensional problems.

In addition to the Lie algebras $su(2), su(1,1)$ and $Sp(4, R)$ the Hamiltonian (1) include two other Lie algebras; $osp(2,1)$ and $osp(2,2)$[32, 33]. The natural step to relate the Hamiltonian (1) and $osp(2,1)$ algebra is to express the Hamiltonian as linear and/or bilinear combinations of the operators of the $osp(2,1)$ algebra. The algebra $su(2)$ can be extended to the $osp(2,1)$ with the operators:

$$V_+ = \sigma_+ a_2, V_- = -\sigma_+ a_1, W_+ = \sigma_- a_1^+, W_- = \sigma_- a_2^+ \tag{45}$$

or by introducing the operators

$$V_+ = \sigma_- a_2, V_- = -\sigma_- a_1, W_+ = \sigma_+ a_1^+, W_- = \sigma_+ a_2^+ \tag{46}$$

The transformation of the operators of the $osp(2,1)$ by the operators S or T gives its one variable 2×2 matrix realizations which are useful for practical applications.

One of the major symmetry group candidates for spin one-half particles is the supergroup $osp(2,2)$ which has four even and four odd generators. Its even generators can be represented by bosons while odd generators are represented by combinations of the fermions and bosons. The superalgebra $osp(2,2)$ might be constructed by extending $su(1,1)$ algebra with the fermionic generators. It is possible to express two set of fermionic generators to extend the $su(1,1)$ algebra to the $osp(2,2)$ algebra. These are given by

$$V_+ = \sigma_- a_2^+, V_- = \sigma_- a_1, W_+ = \sigma_+ a_1^+, W_- = \sigma_+ a_2 \tag{47}$$
$$V_+ = \sigma_+ a_2^+, V_- = \sigma_+ a_1, W_+ = \sigma_- a_1^+, W_- = \sigma_- a_2. \tag{48}$$

Among these let us also mention here, the procedure given here can also related to the nonlinear Lie algebras that have been great deal of interest because of their several significant applications in several branches of physics. Let us illustrate these relation on an example. Consider The effective Hamiltonian, which represents the Jaynes-Cummings model with Kerr nonlinearity, have been expressed as[34]

$$H = \omega a^+ a + \frac{1}{2}\omega_0 \sigma_0 + \kappa(a^+ \sigma_- + a\sigma_+) + \lambda a^+ a a^+ a \tag{49}$$

where κ and λ are coupling constants of the field and atom and coupling constant of the field and Kerr medium respectively. The Hamiltonian (49) can be expressed in terms of the Hamiltonian (1) by an appropriate choice of parameters and the number operator of this structure becomes:

$$N = a_1^+ a_1 + \frac{1}{2}\sigma_0. \tag{50}$$

The invariance algebra of the Hamiltonian (49) is generated by introducing the generators,

$$Y_+ = a_1 \sigma_+, \quad Y_- = a_1^+ \sigma_-, \quad Y_0 = a_1^+ a_1 + \sigma_0 \tag{51}$$

yields the commutation relation:

$$[Y_0, Y_\pm] = \pm Y_\pm, \quad [Y_+, Y_-] = (1 + 2Y_0)(Y_0 - N) - \frac{1}{2} \tag{52}$$

The commutation relation $[Y_+, \quad Y_-]$ is a polynomial in Y_0. Since the deformation is quadratic in Y_0, we have a quadratic algebra. The algebras of type (52) have been considered as deformed $su(2)$ algebra. We can easily express the Hamiltonian (49) in terms of the generators of the deformed $su(2)$ algebra,

$$H = \omega(2N - Y_0) + \omega_0(Y_0 - N) + \kappa(Y_+ + Y_-) + \lambda(2N - Y_0)^2 \tag{53}$$

Note that the number operator N is associated with the conserved quantity of the physical system and it commutes with the generators Y_+, Y_- and Y_0. Consequently, in this article, we have shown various Lie algebraic approaches can be treated in a unified framework.

CONCLUSION

In this paper we have prepared a general method to obtain the solution of two boson and one fermion Hamiltonian. By using either solution of number operator or similarity transformation, we have been able to provide a QES of the various physical Hamiltonians. Furthermore, it has been given that two boson Hamiltonian can be reduced to single variable differential equation in the Bargmann-Fock space. It is also important to mention here that the methods given here can be used to solve higher order differential equations.

The algebras can be realized in several ways. For practical applications, the realizations are given in terms of the boson creation and annihilation operators. In this paper we have discussed the connection between two-boson and one variable differential realizations of the various Lie algebras. The realization in terms of the one variable differential

equation directly leads to the usual Schrödinger formulation. The method given here, can easily be extended to solve the Hamiltonians that include multi-boson or multi fermion-boson systems. We have presented a first step toward the extension of the formulation to obtain solution of the various physical problems. Finally, we have shown that the technique given here provides a unified approach to the various algebraic techniques.

Acknowledgements

* Electronic address: tutunculer@gantep.edu.tr
† Electronic address: koc@gantep.edu.tr

[1] Klimov A B and Sanchez-Soto L L (2000) Pyhs. Rev. A 61 063802
[2] Karassiov V P, Gusey A A and Vinitsky S I (2001) hep-quant/0105152
[3] Gabriel Alvarez and Ramon F Alvarez-Estrada (2001) J. Phys. A: Math. Gen. 34 10045
[4] Alvarez G and Alvarez-Estrada R F (1995) J. Phys. A: Math. Gen. 28 5767
[5] Beckers J, Brihaye Y and Debergh N (1999) J. Phys. A: Math. Gen. 32 2791
[6] Brihaye Y, Kosinski P (1994) J. Math. Phys. **35** 3089
[7] Tjin T (1992) Int. J. Mod. Phys. A7 6175
[8] Abdesselam B, Beckers J, Chakrabarti A and Debergh N (1996) J. Phys. A 29 3075
[9] Sunilkumar V, Bambah B A, Jagannathan R, Panigrahi P K and Srinivasan (2000) J. Opt. B: Quantum Semiclass. Opt. 2 126
[10] Turbiner A V and Ushveridze A G (1987) Phys. Lett. A 126 181
[11] Dolya S N and Zaslavskii O B (2000) J. Phys. A: Math. Gen. 33 L369
[12] Dolya S N and Zaslavskii O B (2001) J. Phys. A: Math. Gen. 34 5955
[13] Eberly J H, Narozhny N B and Sanchez-Mondragon J J (1980) Phys. Rev. Lett. 44 1329
[14] Karassiov V P (1994) J. Phys. A 27 153
[15] Karassiov V P and Klimov A B (1994) Phys. Lett. A 189 43
[16] Delgado J, Luis A, Sánchez-Soto L L and Klimov A B (2000) J. Opt. B: Quantum Semiclass. Opt. 2 33 5.
[17] Klimov A B, Romero J L, Delgado J and Sánchez-Soto L L (2003) J. Opt. B: Quantum Semiclass. Opt. 5 34 6.
[18] Karassiov V P (1998) Phys. Lett. A 238 19
[19] Bandilla A, Drobny G and Jex I (1996) Phys. Rev. A 53 507
[20] Jurco B (1989) J. Math. Phys. 30 1289
[21] Kumar V S, Bambah B A and Jagannathan R (2001) J. Phys. A: Math. Gen 34 8583
[22] Perina J 1991 Quantum Statistics of Linear and Nonlinear Optical Phenomena (Dordrecht: Kluwer) ch 10
[23] Bajer J and Miranowicz A (2000) J. Opt. B: Quantum Semiclass. Opt. 2 L10
[24] Qu Fa, Wei Bao-Hua, Yu K W and Lui Cui-Hong 1996 J. Phys. Condens. Matter 8 2957
[25] Jing-Bo ZU and Xu-Bo ZOU (2001) Chin. Phys. Lett. 18 51
[26] Reik H G, Stülze M E amd Doucha M (1987) J. Phys. A: Math. Gen 20 6327
[27] Koç R, Tütüncüler H, Koca M and Körcük E (2003) Prog. Theor. Phys. 110 399
[28] Alhassid Y, Gürsey F and Iachello F, (1983) Ann. Phys. 148 346
[29] Wybourne B G (1974) " Classical Groups for Physicist" , John Wiley & Sons New York
[30] Alhassid Y and Levine R D 1978 Phys. Rev. A 18 89
[31] Alhassid Y and Koonin S E 1981 Phys. Rev. C 23 1590
[32] Chen Yong-Qing (2000) J. Phys. A: Math. Gen. **33** 8071; (2001) Int. J. Theor. Phys. **40** 1113; (2000) Int. J. Theor. Phys. **39** 2523
[33] Chen Yong-Qing, Xiao-Hui Liu and Xing-Chang Song (1994) Commun. Theor. Phys. 22 123
[34] Buzek V and Jex I (1990) Opt. Cummun. 78 425